TP Méthodes inverses

Problem presentation

We consider the Lorenz equations:

$$\left\{\begin{array}{lcl} \dot{x} & = & 10(y-x) \\ \dot{y} & = & 28x - y - xz \\ \dot{z} & = & xy - \frac{8}{3}z \end{array}\right. \tag{1}$$

where $\dot{x}$ is the derivative of x with respect to time:

$$\dot{x} = x'(t) = \frac{dx}{dt}$$

and similarly for y and z. Thus equation (1) is a non-linear differential system of order 1. In the sequel we will denote the model variable $u = (x, y, z)$, and u_0 the initial condition. The "reference" initial condition of this model is given by:

$$x(0) = x_0 = -4.62, \quad y(0) = y_0 = -6.61, \quad z(0) = z_0 = 17.94$$

We assume that it is unknown: we only know a priori (backgroud) information about it: $u_0^b = (x_0^b, y_0^b, z_0^b) = (-5, -7, 17)$, and we also have observations $u^o(t_i) = (x^o(t_i), y^o(t_i), z^o(t_i))$ at various times t_i (observations are generated from the run of the reference state).
The aim of this work is to do data assimilation to recover the reference state of the system thanks to the background and the observations.

Theoretical work on the continuous model

Question 1. Let $\mathcal{J}$ be the following cost function:

$$\mathcal{J}(u_0) = \int_0^T \|u(t) - u^o(t)\|^2 \, dt$$

Write the tangent model, the adjoint model, and give an expression of the gradient of $\mathcal{J}$.

Question 2. Le us now define the cost function $\mathcal{J}$ with an additional background term:

$$\mathcal{J} = \|u_0 - u^b\|^2 + \int_0^T \|u(t) - u^o(t)\|^2 \, dt$$

Explain what changes in the tangent model, in the adjoint model, and in the gradient of $\mathcal{J}$.

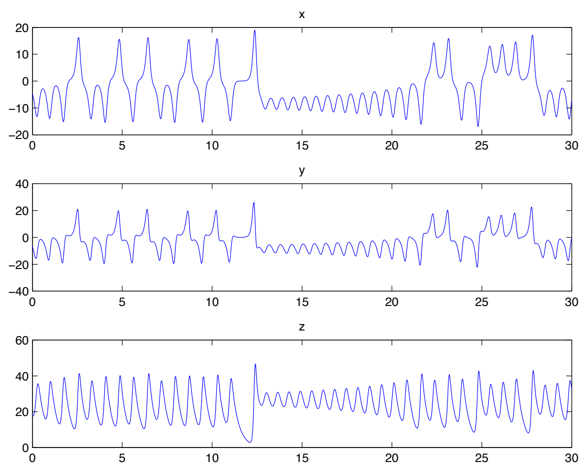
Question 3. Discretise the continuous model and write the Kalman filter algorithm associated to the discrete model you obtained.

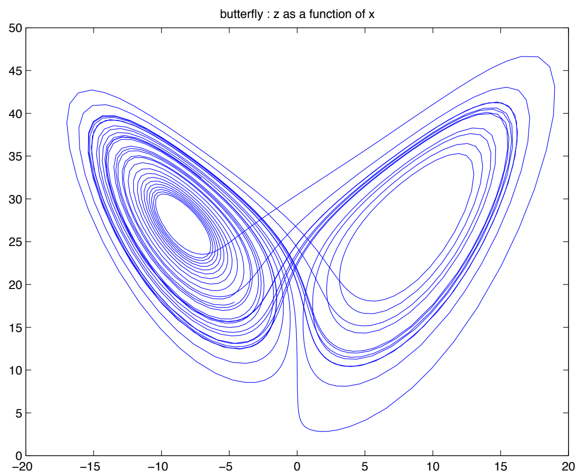
Question 4. Recall the theoretical equivalence with the adjoint method.

Discrete models

Question 5. Write a numerical code solving equation (1). Give details on the numerical method you choose (even if you did not explicitely implement it but used built-in functions) and explain why you choose this particular method.

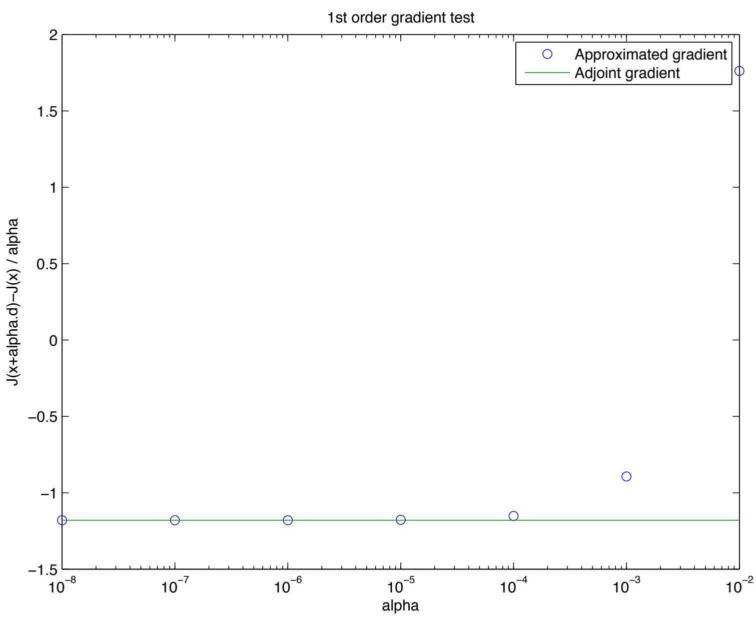
Question 6. In order to validate your model, use the following "reference" initial conditions, with final time 30, time step no smaller than 0.05, and reproduce the two following figures:





Question 7. Write the adjoint code. To do so, you can initialise the model with u_0^{ref} to generate observations. Also write the code computing the gradient of $\mathcal{J}$.

Question 8. Validate the adjoint model using the gradients tests. To do so, you can use $d = u^b - u_0^{\mathrm{ref}}$ as a perturbation direction. Reproduce the following figure, and provide another one for the second order test:



Data assimilation experiments

In this section, you will perform twin experiments. You will first generate the observations with u_0^{ref} as initial conditions. Then you will implement assimilation methods, and you will initialise the algorithms with the bakground.

Question 9. Implement the Kalman filter algorithm, and validate it in the twin experiments framework: initialise the algorithm with the background, use the observations, and check that your analysis u^a is a good approximation of the exact solution u^{ref}. Provide figures to illustrate this point (at least one figure showing the evolution of the reference, the background and the analysis trajectories during the time window $[0, 30]$).

Question 10. Similarly, implement a 4D-Var algorithm for the same twin experiments framework. Similarly provide at least one figure showing reference, background and analysis trajectories over time. Provide also a figure showing the deacrease of the cost function with the iteration number.

Question 11. What happens if observations are not available at every time and space point? Perform some experiments and provide some figures to illustrate them.

Instructions

General instructions:

1. You are asked to write a report and to provide your code sources.
2. Each question must be answered thoroughly and in a detailed fashion (method, algorithm, computations details, figures, tables, and so on).
3. It is strongly advised to produce a written report using LaTeX, see e.g. `http://en.wikibooks.org/wiki/LaTeX` in english, and `http://fr.wikibooks.org/wiki/Programmation_LaTeX` in french.

Coding instructions:

1. Feel free to use built-in numerical methods, e.g. in Matlab:
   - ode45 to solve differential equations
   - fminunc to minimise a function when its gradient is available
2. When asked to reproduce figures, you must provide in your code an easy way to do so, e.g. a script or a program called “Figure1” in your code.
3. More generally, your code must be easy to run, easy to read, and figures must be easy to reproduce. Write plenty of comments in your code and provide instructions to run you code if necessary.

Deadline: December 21st sharp.